Reflection at a Curved Dielectric Interface—Electromagnetic Tunneling

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(Invited Paper)

Abstract—The reflection of a locally plane wave from a curved interface between two nonabsorbing dielectric media is investigated. *Our analysis is applicable to an interface of general shape, defined at each point by the two principal radii of curvature.* When the wave is incident from the denser medium at angles greater than the critical angle it is only partially reflected, due to a form of electromagnetic tunneling. Generalized Fresnel transmission coefficients and an extension of Snell's law are derived to account for this transmission into the less dense medium. Ray tracing can then be applied to determine such phenomena as the bending losses in optical slab waveguides, and the curvature loss of skew rays within straight optical waveguides of circular cross section.

I. INTRODUCTION

We investigate the effect of a curved dielectric interface on the reflection and refraction of plane waves for arbitrary values of the refractive indices on either side of the interface. Our analysis is applicable to interfaces of general shape. When the interface is concave toward the denser medium, we find that total internal reflection is prevented by electromagnetic tunneling. Very simple generalized Fresnel's law and Snell's law are derived to include the effects of tunneling.

The special case of incidence close to the critical angle, when the indices of refraction are nearly equal has been reported previously [1].

We begin in Section II with a brief review of the laws of reflection and refraction for a plane interface, and present the results for a curved interface in Section III. The mathematical derivation is given in Section VI.

II. PLANE INTERFACE (NONABSORBING MEDIA)

Consider a plane wave incident, in the optically denser medium, at a plane interface between two nonabsorbing dielectric media of refractive indices n_1 and $n_2 < n_1$ as shown in Fig. 1. The angles of incidence and transmission relative to the normal are α_i and α_t, respectively. When the wave is incident at an angle that is greater than or equal to the critical angle α_c, it is totally reflected. For $y \geq 0$, the electromagnetic field is evanescent and decays exponentially away from the interface [2].

For $\alpha_i < \alpha_c$, the wave is only partially reflected, since some of the incident light energy is transmitted normally

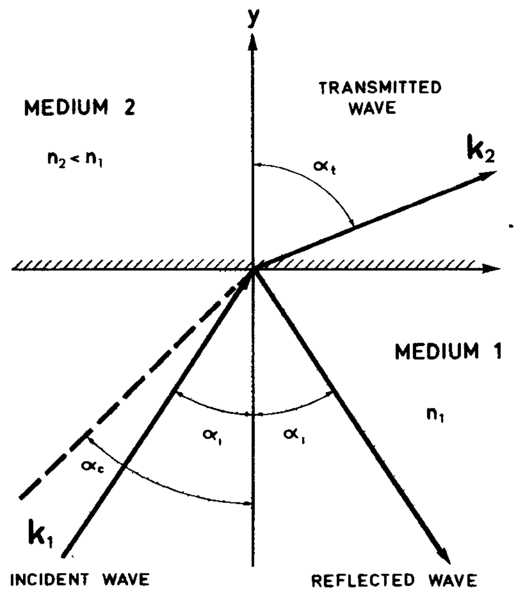

Fig. 1. Plane wave reflection from a plane interface between two nonabsorbing media of indices of refraction n_1 and n_2. The wave vectors in media 1 and 2 are $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$, respectively. The plane of incidence is defined by $\boldsymbol{k}_1$ and the normal (y axis). $\boldsymbol{k}_2$ lies in the plane of incidence. The figure illustrates a wave undergoing refraction. α_c is the critical angle.

to the interface into the less dense medium. The transmitted or refracted wave originates at the interface $y = 0$ at angle α_t given by Snell's law

$$n_1 \sin \alpha_i = n_2 \sin \alpha_t. \qquad (1)$$

The critical angle is that value of α_i for which $\alpha_t = \pi/2$. Thus

$$\sin \alpha_c = n_2/n_1. \qquad (2)$$

We define a power transmission coefficient T as

$$\boxed{T = 1 - \frac{\text{Power of the reflected wave}}{\text{Power of the incident wave}}}. \qquad (3)$$

For the plane interface, T is given by Fresnel's classical expressions $T = T_F$. When $\alpha_i \geq \alpha_c$, $T_F \equiv 0$, and when $0 \leq \alpha_i \leq \alpha_c$, the transmission coefficient depends on the polarization of the incident wave. If the electric vector is parallel to the interface ($E_y = 0$) then $T_F = T_F^E$, where [2]

$$T_F^E = \frac{4\{1 - (\cos \alpha_c/\cos \alpha_i)^2\}^{1/2}}{[1 + \{1 - (\cos \alpha_c/\cos \alpha_i)^2\}^{1/2}]^2}. \qquad (4a)$$

When the magnetic vector is parallel to the interface

Manuscript received March 19, 1974; revised May 28, 1974. This work was supported by the Australian Post Office.
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($H_y = 0$) then $T_F = T_F{}^H$, where [2]

$$T_F{}^H = \frac{4\{1 - (\cos \alpha_c/\cos \alpha_i)^2\}^{1/2}/\sin^2 \alpha_c}{[1 + \{1 - (\cos \alpha_c/\cos \alpha_i)^2\}^{1/2}/\sin^2 \alpha_c]^2}. \quad (4b)$$

In the case of nearly equal refractive indices, $n_1 \cong n_2$, $\sin \alpha_c \cong 1$, and $T_F{}^H \cong T_F{}^E$. On the other hand, when $\alpha_i \cong \alpha_c$ and the refractive indexes are arbitrary

$$T_F{}^E \cong T_F{}^H \sin^2 \alpha_c \cong 4\{1 - (\cos \alpha_c/\cos \alpha_i)^2\}^{1/2} \quad (4c)$$

provided α_c is not too small in the case of $T_F{}^H$.

III. CURVED INTERFACE (SUMMARY OF RESULTS)

We now introduce the modifications to the plane interface results due to curvature. In Fig. 2, ρ is the radius of curvature in the plane of incidence, formed by the normal to the interface (i.e., the y axis) and the wave vector $\boldsymbol{k}_1$ in medium 1. We emphasize that ρ depends on the direction of $\boldsymbol{k}_1$ and the principal radii of curvature that define the interface (see Section IV).

The results of this section assume that 1) the incident radius of curvature is large compared to the wavelength of light in medium 1, and 2) when $\alpha_i > \alpha_c$, $\sin \alpha_i \cong \sin \alpha_c$, and ρ is of the order of the smaller of the two principal radii of curvature. Both of these restrictions are discussed in detail in Section VI, where the derivation of the following results is presented.

A. Origin of the Transmitted Wave

In general, when a plane wave is incident upon a curved interface, it is only partially reflected. The transmitted

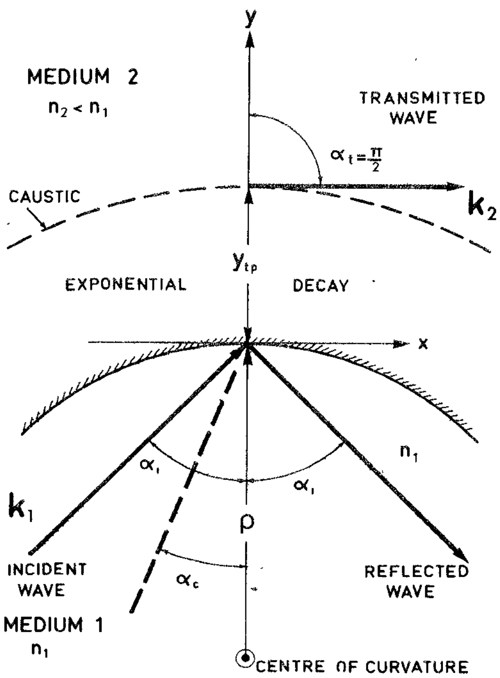

Fig. 2. Reflection from a curved interface between two nonabsorbing media. When $\alpha_i > \alpha_c$, the transmitted wave originates tangent to the caustic at $y = y_{tp} = \rho\{(\sin \alpha_i/\sin \alpha_c) - 1\}$. ρ is the radius of curvature in the plane formed by the normal to the interface and the incident wave direction $\boldsymbol{k}_1$. In general ρ depends on two principal radii of curvature and the direction of $\boldsymbol{k}_1$ as discussed in Section IV.

wave in medium 2 appears to originate at a distance y_{tp} from the interface as shown in Fig. 2, where y_{tp} is given by

$$y_{tp} = \begin{cases} \rho\{(\sin \alpha_i/\sin \alpha_c) - 1\}, & \alpha_i \geq \alpha_c \quad (5a) \\ 0, & \alpha_i \leq \alpha_c. \quad (5b) \end{cases}$$

As the angle of incidence α_i is increased above the critical angle α_c, y_{tp} increases. The position $y = y_{tp}$ specifies the location of the turning point (caustic) between the exponential decay and outgoing wave behavior of the fields. The electromagnetic field is evanescent for $0 \leq y \leq y_{tp}$. *Thus we can view the apparent origin of the transmitted wave at* $y_{tp} > 0$ *as a form of electromagnetic tunneling* [1], [3]. The wave tunnels from the interface at $y = 0$ through the evanescent region to emerge at $y = y_{tp}$. Tunneling arises [2] because the phase velocity of the wave in medium 2, parallel to the curved interface, is less than the velocity of a plane wave in medium 2 for $y < y_{tp}$. At the position $y = y_{tp}$, these two velocities become equal and the field disassociates itself from the interface by radiating into space. This radiation is analogous to that emitted by a relativistic charged particle moving at constant speed on a curved path, i.e., synchroton radiation. Radiation due to refraction is analogous to Cherenkov radiation.

We call the waves with $y_{tp} = 0$, i.e., those with $\alpha_i \leq \alpha_c$, refracting waves. We call waves with $y_{tp} > 0$, i.e., those with $\alpha_i > \alpha_c$, tunneling waves.

B. Angle α_t of the Transmitted Wave at $y = y_{tp}$

After tunneling, the wave emerges *tangent* to the caustic approximately in the plane of the incident wave (see Section VII) as shown in Fig. 2, so that $\alpha_t = \pi/2$ independent of y_{tp}. This is the extension of Snell's law for angles of incidence greater than the critical angle. For $\alpha_i \leq \alpha_c$ the transmission angle is given by (1).

C. Power Transmission Coefficient

The fraction T of incident light power that is transmitted, normal to the interface, into medium 2 is defined by (3). When the interface is curved as in Fig. 2, T is given as

$$\boxed{T = |T_F| C} \quad (6)$$

where T_F is Fresnel's classical transmission coefficient for a plane interface between nonabsorbing media as given by (4). However, unlike (4), T_F in (6) is defined for values of α_i both greater and less than α_c. The curvature factor C is

$$C = \frac{|\mathrm{Ai}\,(\Delta \exp [2\pi i/3])|^{-2}}{4\pi |\Delta|^{1/2}} \quad (7)$$

where

$$\Delta = (k_1\rho/2 \sin^2 \alpha_i)^{2/3} (\cos^2 \alpha_c - \cos^2 \alpha_i) \quad (8)$$

$$k_1 = 2\pi n_1/\lambda \quad (9)$$

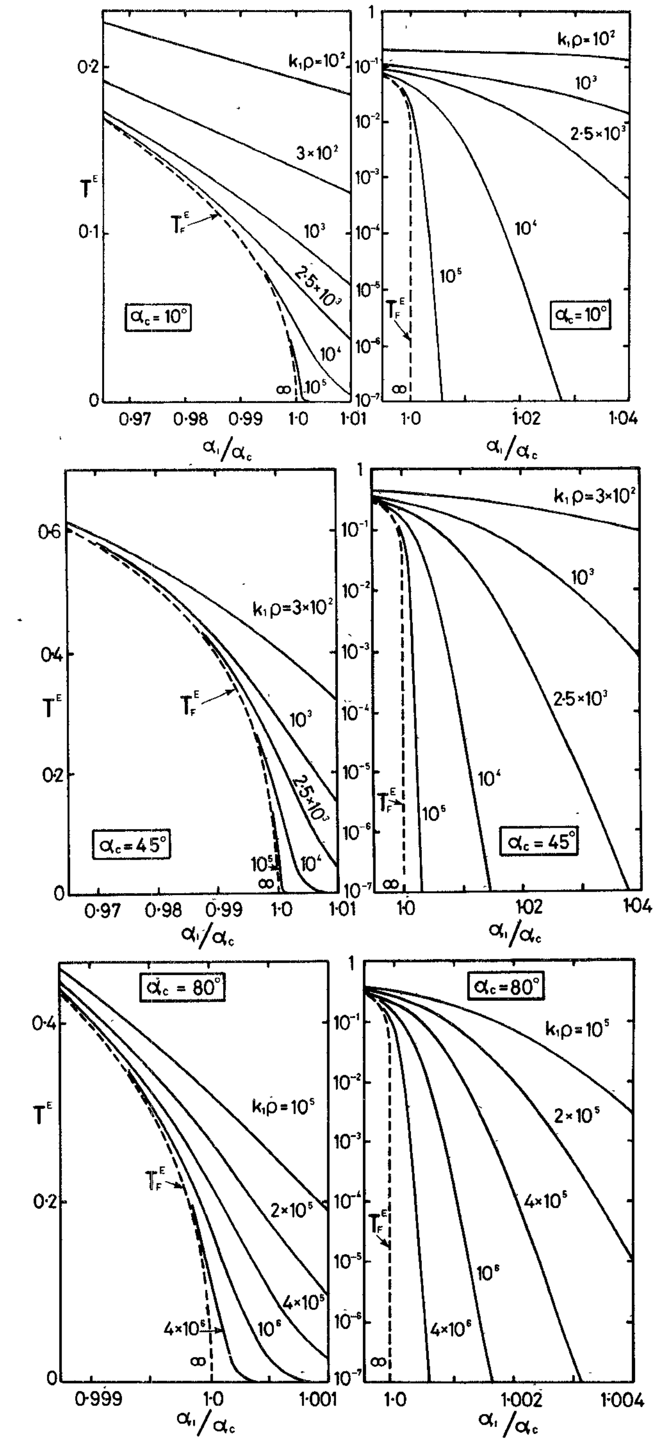

Fig. 3. The power transmission coefficient T^E as defined by (3) for the electric vector $\boldsymbol{E}$ parallel to the interface in Fig. 2. The numerical results are found from (6) for $\alpha_c = 10°$, $45°$, and $80°$. As $k_1\rho \to \infty$, T^E approaches the classical Fresnel coefficient $T_F{}^E$.

and λ is the wavelength in vacuum. The modulus of the Airy function $|\,\text{Ai}\,(\Delta \exp[2\pi i/3])\,|^{-2}$ is a smooth decreasing function of Δ. In the Appendix we give useful asymptotic forms for $|\,\text{Ai}\,(\Delta \exp[2\pi i/3])\,|$. Further details and tables of Airy functions are found in [4].

For angles of incidence close to the critical angle, a good approximation to T is to set $\alpha_i \cong \alpha_c$ in (6). Then, from (4c), (6), (7), and (8)

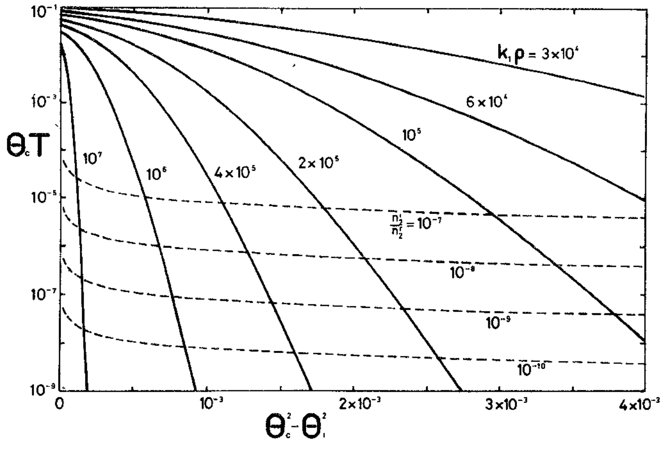

Fig. 4. The power transmission coefficient when $n_1 \cong n_2$, i.e., $\alpha_c \cong \pi/2$ and $T^E \cong T^H$. $\theta_c = (\pi/2) - \alpha_c$ and $\theta_i = (\pi/2) - \alpha_i$. Solid curves represent values of $\theta_c T$ due to curvature, and dashed curves represent $\theta_c T$ due to material absorptions. Where the two sets of curves intersect, the points represent equal transmission losses due to curvature and absorption due to reflection for the parametric values of $k_1\rho$ and $n_2{}^i/n_2{}^r$, respectively.

$$T^E \cong T^H \sin^2 \alpha_c \cong \frac{1}{\pi \cos \alpha_c}\left(\frac{2 \sin^2 \alpha_c}{k_1\rho}\right)^{1/3} \cdot |\,\text{Ai}\,(\Delta \exp[2\pi i/3])\,|^{-2} \quad (10)$$

where the superscripts E and H represent the $E_y = 0$ and $H_y = 0$ cases, respectively, and Δ is given by (8).

These results are formally restricted to the two conditions above. In more precise mathematical terms these are 1) $k_1\rho \gg \sin^{-4} \alpha_c \cos^{-3} \alpha_i$ and 2) $y_{tp} \ll \rho_s$, where ρ_s is the smaller of the two principal radii of curvature; however, they are often more restrictive than necessary (see Section VII).

The expressions for T given by (6) and (10) are uniformly valid on either side of the critical angle. They simplify for the several cases of practical interest to be discussed next.

1) Angle of Incidence α_i Less Than the Critical Angle α_c: When $\alpha_i \leq \alpha_c$, Fresnel's classical expression for T is given by (4). However, we see from the asymptotic form of $|\,\text{Ai}\,|^{-2}$ given by (A7) in the Appendix that $C \sim 1$, i.e., (6) reduces to (4), only when $-\Delta \gg 1$. In other words, when the interface is curved, Fresnel's laws for $\alpha_i \leq \alpha_c$ fail for angles of incidence too close to the critical angle.

2) Incidence at the Critical Angle: When $\alpha_i = \alpha_c$, Fresnel's law gives $T = 0$. However, from (6) or (10) and (A10) of the Appendix, we find that

$$\boxed{T^E = T^H \sin^2 \alpha_c \cong 3.182\,(\sin^2 \alpha_c/k_1\rho)^{1/3}/\cos \alpha_c.} \quad (11)$$

The reader is cautioned from the erroneous conclusion that T is unbounded as $\alpha_c \to 0$ or $\pi/2$. Due to restriction 1) discussed above, $T \ll 1$ when $\alpha_i = \alpha_c$.

3) Incidence Greater Than the Critical Angle. When $\alpha_i > \alpha_c$ and not too close to α_c, in particular, when $\Delta \gg 1$ but not so large as to violate our second restriction, we find from (A4) in the Appendix that

$$\boxed{C \sim \exp\left[-\tfrac{4}{3}\Delta^{3/2}\right]} \qquad (12)$$

which is an excellent approximation for determining curvature losses in nearly all cases of practical interest.

Optical devices often have similar refractive indices, i.e., $n_1 \cong n_2$. In this case we define $\theta_i = \pi/2 - \alpha_i$ and $\theta_c = \pi/2 - \alpha_c$ whence $\theta_i, \theta_c \ll 1$. Thus T from (4a), (6), and (12) simplifies to

$$T^E \cong T^H \cong 4(\theta_i/\theta_c)\{1 - \theta_i^2/\theta_c^2\}^{1/2} \cdot \exp\left[-\tfrac{2}{3}k_1\rho(\theta_c^2 - \theta_i^2)^{3/2}\right]. \qquad (13)$$

Note that we have taken the modulus of T_F^E which for $\alpha_i > \alpha_c$ has an imaginary numerator and a complex denominator.

The restriction $\sin \alpha_i \cong \sin \alpha_c$ implies $\alpha_i \cong \alpha_c$ provided n_1 and n_2 are sufficiently different. In this case, using (4c), (6), and (12), T becomes:

$$T^E \cong 4\left(\frac{\cos^2 \alpha_c}{\cos^2 \alpha_i} - 1\right)^{1/2} \cdot \exp\left\{-\frac{2}{3}\frac{k_1\rho}{\sin^2 \alpha_i}(\cos^2 \alpha_c - \cos^2 \alpha_i)^{3/2}\right\} \qquad (14)$$
$$\cong T^H \sin^2 \alpha_c.$$

This is valid unless $n_1 \cong n_2$. Then (13) is a better approximation.

D. Numerical Results for T

In Fig. 3 numerical results are presented for T when $\alpha_c = 10°$, $45°$, and $80°$, respectively. When $n_1 \cong n_2$, $\theta_c T$ (where T is given by (6)) depends only on $\theta_i^2 - \theta_c^2$ and $k_1\rho$, where $\theta = (\pi/2) - \alpha$. Thus we can plot curves on one graph that are valid for all $\theta_c \ll 1$. This is done in Fig. 4 with the solid curves.

E. Interface with Convex Curvature

Our results apply only to the concave curvature illustrated in Fig. 2. There is no tunneling when the curvature is convex, thus Fresnel's coefficients given by (4) provide a suitable approximation when $k_1\rho \gg 1$.

F. Case of $n_2 > n_1$

When the index of refraction of medium 2 is greater than medium 1, there is no tunneling and Fresnel's coefficients (4) are again a suitable approximation.

IV. DISCUSSION OF RESULTS

We have described reflection from a curved dielectric interface. Our results apply to a generalized interface defined by the principal radii of curvature ρ_x and ρ_z as in Fig. 5. For this case ρ, the radius of curvature in the plane of incidence, is given by

$$\boxed{\rho = \frac{\rho_x \rho_z \sin^2 \alpha_i}{\rho_x \cos^2 \theta_z + \rho_z \cos^2 \theta_x}} \qquad (15)$$

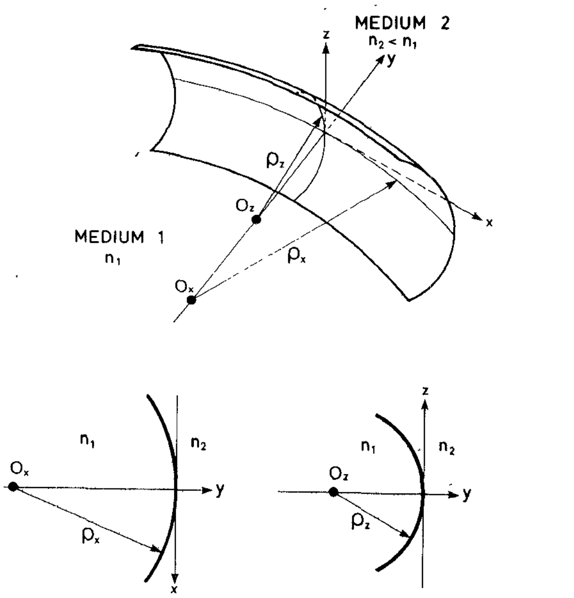

Fig. 5. A curved interface between two dielectric media defined by the principal radii of curvature ρ_x, ρ_z. The interface is concave towards n_1.

where θ_x and θ_z are the inclinations of the wave vector $\boldsymbol{k}_1$ to the x and z axes, respectively.

The results are valid for local plane wave reflection at an interface between two *semi-infinite* dielectric media. In general, other effects must be considered when attempting to use a ray description of wave propagation, e.g., the lateral shift caused by either a finite beam size [5] or the confinement of light within an optical waveguide [1], [6].

The remarkable feature of the power transmission coefficient T as given by (6) is that it depends only on the radius of curvature ρ in the plane of incidence and the inclination α_i to the normal. This simplicity is a consequence of the restriction $y_{ip} \ll \rho_x, \rho_y$ requiring $\sin \alpha_i \cong \sin \alpha_c$ as shown in Sections VI and VII. If it were true for all angles α_i, T would be zero only when $\rho = \infty$. However, $T = 0$ for all skew rays within a *cylinder* that are inclined to the cylinder axis at angles less than $(\pi/2) - \alpha_c$ [1], [7], [8]. For example, if we let $\rho_z = \infty$ in Fig. 2, then from (15), $\rho = \rho_x \sin^2 \alpha_i / \cos^2 \theta_x$. All rays with $\theta_z \le (\pi/2 - \alpha_c)$ have $T = 0$ but $\rho \ne \infty$ unless $\theta_x = \pi/2$. Nevertheless, comparing the exact T for cylinders with our approximate T shows that (6) is an excellent approximation for *all* rays with $\alpha_i \gtrsim \alpha_c$ [9]. *In nearly all cases of practical interest,* (6) *and its simple interpretation allow for a satisfactory description of reflection from a curved interface.*

Although the curvature loss is small for one reflection, it is significant for multiple reflections. For example, a substantial amount of light leaks from the straight, multimode circular optical waveguide due to its curved cross section [9]. These losses are easily accounted for by ray tracing, using our generalized Fresnel transmission coefficient [9]. Curvature losses explain the difference between leaky modes on circular and slab waveguides [7], [10], [11] and also help to explain why there is no trapped energy within a finite dielectric structure of higher index of refraction than its surround [12]. Numerous applications of the generalized Fresnel's laws are discussed in [1] for the special case of $n_1 \cong n_2$.

V. EFFECT OF MATERIAL ABSORPTION ON REFLECTION AT A PLANE INTERFACE

Total reflection at a *plane interface* is possible only if both media are nonabsorbing. Since the partial reflection due to curvature is a small effect for *one reflection*, it is necessary to examine when it is masked by the partial reflection caused by material absorption.

When both media are absorbing, the Fresnel power coefficients defined by (3) for a plane interface are

$$T^E = 4(\cos\alpha_i/\cos^2\alpha_c)\ \mathrm{Re}\{(n_2/n_1)^2 - \sin^2\alpha_i\}^{1/2} \qquad (16a)$$

$$T^H = \left(\frac{4\cos\alpha_i\sin^4\alpha_c}{\cos^2\alpha_c}\right) \frac{\mathrm{Re}\{(n_1/n_2)^2[(n_2/n_1)^2 - \sin^2\alpha_i]^{1/2}\}}{\{\sin^2\alpha_i - \sin^2\alpha_c\cos^2\alpha_i\}} \qquad (16b)$$

provided that $|\cos^2\alpha_i - \cos^2\alpha_c| \gg |(n_2^i/n_2^r) - (n_1^i/n_1^r)|$, where Re is the real part and $n_1 = n_1^r - in_1^i$, $n_2 = n_2^r - in_2^i$ (assuming a time dependence $\exp(-i\omega t)$).

For $n_1^r \cong n_2^r$, $\alpha_c \cong \pi/2$ and (16) simplifies

$$T^E \cong T^H \cong 4(\theta_i/\theta_c^2)\ \mathrm{Re}[\theta_i^2 - \theta_c^2 + 2i\{(n_2^i/n_2^r) - n_1^i/n_1^r)\}]^{1/2} \qquad (17)$$

where $\theta_i = (\pi/2) - \alpha_i$ and $\theta_c = (\pi/2) - \alpha_c$. We are interested in the case of small absorption. Thus, when $\theta_i < \theta_c$,

$$\boxed{T^E \cong T^H \cong 4\left(\frac{\theta_i}{\theta_c^2}\right)\frac{\{(n_2^i/n_2^r) - (n_1^i/n_1^r)\}}{(\theta_c^2 - \theta_i^2)^{1/2}}} \qquad (18)$$

provided $\theta_c^2 - \theta_i^2 \gg |(n_2^i/n_2^r) - (n_1^i/n_1^r)| \ll 1$. In the case when $n_2^i/n_2^r < n_1^i/n_1^r$, T is indeed negative, corresponding to an inflow of energy into medium 1 from the evanescent field in medium 2.

Numerical values of (17) are plotted as dashed curves in Fig. 4 for the case $\theta_i \cong \theta_c$ and $n_1^i = 0$. Equation (18) is highly accurate throughout the entire region depicted.

Absorption loss is often expressed in dB/km instead of n_2^i, where $n_2^i = (\lambda/4\pi 10^4 \ln_{10} e)$ dB/km and λ is the wavelength in vacuum.

As a practical example, we consider parameters typical of the materials used for an optical fiber [13]; $n_1^r \cong 1.551$, $n_2^r \cong 1.4648$, and $\lambda = 0.633\ \mu$m. With these values we find that $n_2^i/n_2^r \cong 0.7918 \times 10^{-11}$ dB/km. Thus from Fig. 4 the dashed curves for $n_2^i/n_2^r = 10^{-7}, 10^{-8}, 10^{-9}$, and 10^{-10} correspond to attenuations of 12 628, 1263, 126, 12.6 dB/km, respectively. The greatest absorption is representative of a glass far too lossy as a cladding for communication optical waveguides. Nevertheless, Fig. 4 shows clearly that, for any given $k_1\rho$, reflection losses due to curvature far exceed those due to absorption for incidence near to the critical angle.

The reader is reminded that we have been discussing the effects of the absorbing media on reflection and not absorption due to the path length of the ray.

VI. MATHEMATICAL DERIVATION

Here we present an analysis of reflection from the interface between two dielectric media shown in Fig. 5. This is accomplished by considering the curved boundary as a perturbation of a plane interface so that the analysis is a modification of that used to derive Fresnel's classical laws. We assume $k_1\rho \gg 1$, so that the fields exhibit local plane wave characteristics. Accordingly the field in the denser medium 1 can be approximated by an incoming and an outgoing plane wave. The wave number k_1 is defined by (9) and the radius of curvature in the plane of incidence ρ by (15). As we show below, *the effect of curvature is accounted for only in the fields of medium 2 and not by the boundary conditions.*

For a *plane* interface, the magnitudes k_1 and k_2 of the wave vectors $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ in medium 1 and 2 are related as

$$k_1^2 = k_x^2 + k_{y1}^2 + k_z^2 = (2\pi n_1/\lambda)^2 \qquad (19a)$$

$$k_2^2 = k_x^2 + k_{y2}^2 + k_z^2 = (2\pi n_2/\lambda)^2 \qquad (19b)$$

where

$$k_x = k_1\cos\theta_x \qquad k_{y1} = k_1\cos\alpha_i \qquad k_z = k_1\cos\theta_z \qquad (20a)$$

$$k_{y2} = (k_2^2 - k_1^2 + k_{y1}^2)^{1/2} = k_1(\sin^2\alpha_c - \sin^2\alpha_i)^{1/2} \qquad (20b)$$

$$\cos^2\theta_x + \cos^2\alpha_i + \cos^2\theta_z = 1. \qquad (20c)$$

Angles θ_x and θ_z are the inclinations to the x and z axes, respectively, and α_i is the inclination to the y axis or normal. The cartesian components $\psi(x,y,z)$ of the vector fields satisfy the scalar wave equation for a homogeneous media. Thus, in medium 2, a solution is

$$\psi = e(y)\exp(ik_xx + ik_zz) \qquad (21a)$$

where

$$(d^2/dy^2) + k_{y2}^2)\, e(y) = 0. \qquad (21b)$$

So far we have only been considering the plane interface. An exact solution of the fields for the curved interface must have variations of the forms $\exp(il_x\phi_x)$ and $\exp(il_z\phi_z)$, where angles ϕ_x and ϕ_z are azimuthal angles in the $x - y$ and $y - z$ planes, referred to origins O_x and O_z at the centers of curvature (Fig. 5). To account for curvature, we assume that k_x and k_z are functions of y to satisfy

$$\exp[ik_x(y)x] \cong \exp[il_x\phi_x] \qquad \exp[ik_z(y)z] \cong \exp[il_z\phi_z] \qquad (22)$$

in the neighborhood of the interface. Since $x \cong (\rho_x + y)\phi_x$ and $z \cong (\rho_z + y)\phi_z$, we have

$$k_x(y) \cong l_x/(\rho_x + y) \qquad k_z(y) \cong l_z/(\rho_z + y). \qquad (23)$$

The azimuthal parameters l_x, l_z are defined by observing that $k_x(0) \cong k_x$ and $k_z(0) \cong k_z$ on the interface. This leads to

$$l_x \cong \rho_x k_x \qquad l_z \cong \rho_z k_z. \tag{24}$$

If we restrict the analysis to $y \ll \rho_x, \rho_z$, then

$$k_x^2(y) + k_z^2(y) = (k_x^2 + k_z^2)(1 - 2y/\rho) \tag{25}$$

where ρ is given by (15).

The electromagnetic fields in medium 2 are found by replacing k_{y2}^2 in (21b) by

$$k_{y2}^2(y) = k_2^2 - k_x^2(y) - k_z^2(y) \tag{26a}$$

$$= k_1^2 (\sin^2 \alpha_c - \sin^2 \alpha_i + 2y \sin^2 \alpha_i/\rho). \tag{26b}$$

After a change of variables this leads to Airy's equation

$$(d^2/d\xi^2 - \xi)e(\xi) = 0 \tag{27}$$

where ξ is dimensionless

$$\xi(y) = -\{2(k_x^2 + k_z^2)/\rho\}^{-2/3}\{k_{y2}^2 + 2(k_x^2 + k_z^2)y/\rho\} \tag{28a}$$

$$= \{2 \sin^2 \alpha_i/k_1\rho\}^{-2/3}\{\sin^2 \alpha_i - \sin^2 \alpha_c - 2y \sin^2 \alpha_i/\rho\}. \tag{28b}$$

The Airy function solutions of (27) can be thought of best as connecting expressions between oscillatory and evanescent (exponential) behavior. They give the field dependence in the neighborhood of the caustic or turning point of a large class of solutions to the scalar wave equation, e.g., the Bessel functions in the region where order equals argument.

The fields must be outward going waves as $y \to \infty$ and also satisfy the boundary conditions at the dielectric interface. The appropriate linear combination of Airy functions required to represent outward going waves is Ai $(\xi \exp [2\pi i/3])$. This function is discussed in the Appendix.

The amplitude coefficients of the fields are found by satisfying the boundary conditions at the dielectric interface. In general, TE ($E_y = 0$) and TM ($H_y = 0$) type waves couple at a nonplanar interface. However, our perturbation method involves boundary conditions that are the same as those for a plane interface. We have included the effect of curvature in the scalar wave functions $e(\xi)$ found from (27). Further, subject to our approximations, $k_1\rho \gg 1$ and $y \ll \rho_x, \rho_z$, the $e(\xi)$ *depending on one radius of curvature* ρ *only*. Thus reflection from an arbitrary surface defined by two principal radii of curvature is equivalent to reflection from a cylindrical surface of radius ρ, when the incident wave is in the cylinder cross section, i.e., has no axial component. For this case TM and TE modes do not couple and we can solve for the amplitude coefficients with either $E_y = 0$ or $H_y = 0$.

We begin with $E_y = 0$, so that $\boldsymbol{E}$ is parallel to the interface. Then, in medium 1 (omitting the time dependence $\exp [-i\omega t]$),

$$E_z^{(1)} = (a \exp [ik_{y1}y] + b \exp [-ik_{y1}y]) \cdot \exp [i(k_x x + k_z z)] \tag{29a}$$

$$H_x^{(1)} = \frac{n_1 k_{y1}}{k_1} (a \exp [ik_{y1}y] - b \exp [-ik_{y1}y]) \cdot \exp [i(k_x x + k_z z)] \tag{29b}$$

where a and b are the amplitudes of the z components of the incident and reflected electric fields. In medium 2 for $0 \le y \ll \rho_x, \rho_z$ we have

$$E_z^{(2)} = c \text{ Ai } (\xi \exp [2\pi i/3]) \exp [i(k_x x + k_z z)] \tag{30a}$$

$$H_x^{(2)} = \frac{-in_2 c}{k_2} \frac{d}{dy} \text{Ai } (\xi \exp [2\pi i/3]) \exp [i(k_x x + k_z z)], \tag{30b}$$

where c is a constant. Continuity of E_z and H_x at $y = 0$ leads to

$$\frac{b}{a} = \frac{\psi - 1}{\psi + 1} \tag{31}$$

where

$$\psi = -i \text{ Ai } (\Delta \exp [2\pi i/3])/\gamma \text{ Ai}' (\Delta \exp [2\pi i/3]) \tag{32}$$

$$\gamma = \{2(k_x^2 + k_z^2)/\rho\}^{1/3}/k_{y1} \tag{33a}$$

$$= (2 \sin^2 \alpha_i/k_1\rho)^{1/3}/\cos \alpha_i \tag{33b}$$

$$\Delta = \xi(y = 0). \tag{34}$$

Prime denotes differentiation with respect to Δ. The power transmission coefficient defined by (3) is

$$T^E = 1 - |b/a|^2 \tag{35a}$$

$$= 4 \text{ Re } \psi/(|\psi|^2 + 2 \text{ Re } \psi + 1) \tag{35b}$$

where Re ψ is the real part of ψ given as

$$\text{Re } \psi = (\psi + \psi^*)/2 = |\text{Ai } (\Delta \exp [2\pi i/3])|^{-2}/4\pi\gamma. \tag{36}$$

The asterisk denotes the complex conjugate and we have used the Wronskian (A11). The modulus of ψ is simply $|\psi| = (1/\gamma) \,|\text{Ai } (\Delta \exp 2\pi i/3)/\text{Ai}' (\Delta \exp [2\pi i/3])|$.

Generalized Fresnel's Laws and Power Transmission Coefficients

We can find an excellent approximation to T^E by examining its limiting forms. When $\Delta \gg 1$, (35b), (36), (A3), and (A4) lead to

$$T^E \cong |T_F^E| \exp (-4\Delta^{3/2}/3). \tag{37a}$$

When $-\Delta \gg 1$, (35b), (36), (A6), and (A7) lead to

$$T^E \cong T_F^E = |T_F^E| \tag{37b}$$

where T_F^E is Fresnel's classical power transmission coefficient given by (4a) or equivalently as

$$T_F^E = 4k_{y1}k_{y2}/(k_{y1} + k_{y2})^2. \tag{38}$$

We then note that the limiting forms (37) are equivalent to the limiting forms of

$$T^E = |F_F{}^E| C \tag{39}$$

where C is defined by (7) and Δ by (8) or (34) and (28). This suggests that T^E given by (39) may be valid for all Δ. The greatest departure of (39) from (35b) is at $\Delta = 0$, i.e., at $\alpha_i = \alpha_c$. Then, from (4c), (39), and (A10), our approximation (39) is

$$T_{\mathrm{APRX}}{}^E = 3.182\,(\sin^2 \alpha_c / k_1\rho)^{1/3} / \cos \alpha_c. \tag{40}$$

The more exact expression given by (35b) is

$$T^E = T_{\mathrm{APRX}}{}^E / \{1 + (T_{\mathrm{APRX}}{}^E/2) + (T_{\mathrm{APRX}}{}^E/2\sqrt{3})^2\}. \tag{41}$$

Thus (39) is valid for all Δ when at $\alpha_i = \alpha_c$, $T_{\mathrm{APRX}}{}^E \ll 1$. Since, by an earlier assumption, $k_1\rho \gg 1$, $T_{\mathrm{APRX}}{}^E$ is small unless $\alpha_c \cong \pi/2$. Therefore, from (40), we conclude that $k_1\rho \gg \cos^{-3} \alpha_c$ is necessary for (39) to be valid.

In the next section we show that the restriction $k_1\rho \gg \cos^{-3} \alpha_i$ *is necessary to consider plane wave incidence so that* T^E *given by* (39) *is no less accurate than* T^E *given by* (35).

When $H_y = 0$, the above procedure leads to $T^H = |T_F{}^H| C$, provided $k_1\rho \gg \sin^{-4} \alpha_c \cos^{-3} \alpha_c$, where $T_F{}^H$ is given by (4b) or equivalently by

$$T_F{}^H = 4n_1{}^2 n_2{}^2 k_{y1} k_{y2} / \{n_2{}^2 k_{y1} + n_1{}^2 k_{y2}\}^2. \tag{42}$$

The constraint on $k_1\rho$ again arises from requiring the equality of T^H and $T_{\mathrm{APRX}}{}^H$ at $\alpha_i = \alpha_c$. *T^H corresponding to (35) is no less accurate for plane wave incidence than* $T^H = |T_F{}^H| C$ *corresponding to* (39), *unless the condition* $k_1\rho \gg \sin^{-4} \alpha_c$ *is not satisfied. This condition is only a restriction on very small angles* α_c.

VII. ASSUMPTIONS IN THE ANALYSIS

A. $k_1\rho \gg \sin^{-4} \alpha_c \cos^{-3} \alpha_i$

We initially assumed that $k_1\rho \gg 1$ to ensure that the fields are locally plane, but how large must $k_1\rho$ be relative to α_i? This can be determined when it is recalled that we reduced the problem to a wave incident, with no axial component, at a cylindrical boundary. The exact solution for this problem has fields of the form $J_l(k_1\rho) \exp(il\phi)$ at the interface in the denser region, where J_l is a Bessel function of the first kind of order l and ϕ is the azimuthal angle. Only when the Debye condition

$$k_1\rho \gg 1 \quad \text{and} \quad k_1\rho - l \gg l^{1/3} \tag{43}$$

holds can $J_l(k_1\rho) \exp(il\phi)$ be decomposed into an incoming and an outgoing local plane wave at each point (ρ,ϕ) on the interface [7]. Combining these restrictions with the relation [7], $l = k_1\rho \sin \alpha_i$, we deduce

$$1 - \sin \alpha_i \gg \{\sin \alpha_i / (k_1\rho)^2\}^{1/3} \tag{44}$$

which is always satisfied when $k_1\rho \gg 1$ unless α_i is close to $\pi/2$, for which case we find (44) compatible with $k_1\rho \gg (\cos \alpha_i)^{-3}$. This is consistent with the approximation used to obtain (39) from (35). *Thus (39) is the appropriate generalized Fresnel's power transmission coefficient for* T^E.

Since the expression for T^H required $k_1\rho \gg \sin^{-4} \alpha_c \cdot \cos^{-3} \alpha_c$, the Debye condition alone is not sufficient for its validity and our simple expression $T^H = |T_F{}^H| C$ is inaccurate for angles α_c too small to satisfy $k_1\rho \gg \sin^{-4} \alpha_c$. Fortunately, this is a case of little practical interest.

B. $y_{tp} \ll \rho_x,\rho_z$

W assumed that $y \ll \rho_x,\rho_z$ in the derivation of Airy's differential equation (27) so that the scalar wave functions depended on ρ only. From studying the fields in medium 2 we see that there is a caustic or turning point at a position $y = y_{tp} \geq 0$ determined from $\xi(y_{tp}) = 0$ in (28a), i.e.,

$$y_{tp} = -\rho k_{y2}{}^2 / 2(k_x{}^2 + k_z{}^2) = (\rho/2)\{1 - \sin^2 \alpha_c / \sin^2 \alpha_i\} \tag{45}$$

for $\alpha_i > \alpha_c$, and $y_{tp} = 0$ for $\alpha_i \leq \alpha_c$. The fields are evanescent for $0 < y < y_{tp}$ and oscillatory for $y > y_{tp}$. In order to satisfy the outgoing wave condition our wave functions must be valid for $y > y_{tp}$, although not necessarily for $y \gg y_{tp}$. However, we are restricted to $y \ll \rho_x,\rho_z$. Assuming that $\rho_x \leq \rho_z$, we must therefore have $y_{tp} \ll \rho_x$. (This is satisfied automatically when $\alpha_i \leq \alpha_c$ since $y_{tp} \equiv 0$). Hence we require from (45), $(\rho/2\rho_x)\{1 - \sin^2 \alpha_c / \sin^2 \alpha_i\} \ll 1$ which is satisfied when $\sin \alpha_c \cong \sin \alpha_i$ unless (ρ/ρ_x) is enormous. Thus when $\sin \alpha_i \cong \sin \alpha_c$ in (45), y_{tp} is given by (5).

The reader familiar with asymptotic methods for the solution of differential equations may question the need for the assumption $y \ll \rho_x,\rho_z$, or equivalently $\sin \alpha_i \cong \sin \alpha_c$. Instead, the Wentzel–Kramer–Brillouin (WKB) method of solution could have been used directly, although the resulting integral in the exponential function can in general only be evaluated when $\sin \alpha_i \cong \sin \alpha_c$. One case where it can be evaluated exactly is for the cylindrical interface. However, we find that such solutions are valid only when $\sin \alpha_i \cong \sin \alpha_c$. In other words, the restriction $\sin \alpha_i \cong \sin \alpha_c$, necessary when $\alpha_i > \alpha_c$, is due to a fundamental limitation of the perturbation method presented here.

In the analysis we treated TE and TM waves separately. In general, TM and TE waves couple. The situation is then analogous to reflection from an anisotropic slab with effective refractive indices n_x,n_z along the principal axes. The effect of the anisotropic behavior can be seen in our present example by determining the direction $\boldsymbol{k}_2$ of the transmitted wave at $y = y_{tp}$. From (26a), $k_{y2}(y_{tp}) = 0$ so that the wave is tangent to the caustic. However, unless $y_{tp} \ll \rho_x,\rho_z$, we observe that the direction of $\boldsymbol{k}_2$, given by (23), is not in the incident plane, i.e., the wave direction has been twisted in the tunneling process [14]. Thus, when $y_{tp} \ll \rho_x, \rho_z$, we are justified in separating the TM and TE waves.

In conclusion, the concept of a generalized Fresnel's law, that depends only on the radius of curvature in the plane of incidence, fails unless $\sin \alpha_i \cong \sin \alpha_c$, when $\alpha_i > \alpha_c$.

APPENDIX
AIRY FUNCTIONS [4]

The linearly independent solutions of (27) are $\mathrm{Ai}(\xi)$ and $\mathrm{Bi}(\xi)$, and the appropriate linear combinations for our analysis are

$$\mathrm{Ai}\,(\xi \exp [\pm 2\pi i/3]) = (\exp [\pm i\pi/3]/2)\{\mathrm{Ai}\,(\xi) \mp iBi(\xi)\} \quad \text{(A1)}$$

where $\mathrm{Ai}\,(\xi \exp [-2\pi i/3])$ is the complex conjugate of $\mathrm{Ai}\,(\xi \exp [2\pi i/3])$.

1) $\xi \gg 1$ (applicable when $\alpha_i > \alpha_c$)

$$\mathrm{Ai}\,(\xi \exp [2\pi i/3]) \cong \exp [-i\pi/6] \exp [(2/3)\xi^{3/2}]/2\pi^{1/2}\xi^{1/4} \quad \text{(A2)}$$

$$|\mathrm{Ai}'\,(\xi \exp [2\pi i/3])/\mathrm{Ai}\,(\xi \exp [2\pi i/3])| \cong \xi^{1/2} \quad \text{(A3)}$$

$$|\mathrm{Ai}\,(\xi \exp [2\pi i/3])|^{-2} \cong 4\pi\xi^{1/2} \exp [-(4/3)\xi^{3/2}]. \quad \text{(A4)}$$

2) $-\xi \gg 1$ (applicable when $\alpha_i < \alpha_c$)

$$\mathrm{Ai}\,(-\xi \exp [2\pi i/3]) \cong \exp [i\pi/12] \exp [i(2/3)\xi^{3/2}]/2\pi^{1/2}\xi^{1/4} \quad \text{(A5)}$$

$$|\mathrm{Ai}'\,(-\xi \exp [2\pi i/3])/\mathrm{Ai}\,(-\xi \exp [2\pi i/3])| \cong \xi^{1/2} \quad \text{(A6)}$$

$$|\mathrm{Ai}\,(-\xi \exp [2\pi i/3])|^{-2} \cong 4\pi\xi^{1/2}. \quad \text{(A7)}$$

3) $\xi = 0$ (applicable when $\alpha_i = \alpha_c$)

$$\mathrm{Ai}\,(0 \cdot \exp [2\pi i/3]) = 3^{-2/3}/\Gamma(\tfrac{2}{3}) = 0.3550 \quad \text{(A8)}$$

$$|\mathrm{Ai}'\,(0 \cdot \exp [2\pi i/3])/\mathrm{Ai}\,(0 \cdot \exp [2\pi i/3])| = 3^{1/2}\Gamma(\tfrac{2}{3})/\Gamma(\tfrac{1}{3}) = 0.7290 \quad \text{(A9)}$$

$$|\mathrm{Ai}\,(0 \cdot \exp [2\pi i/3])|^{-2} = 3^{4/3}\Gamma(\tfrac{2}{3})^2 = 7.9337. \quad \text{(A10)}$$

4) Wronskian

$$\mathrm{Ai}\,(\xi \exp [2\pi i/3])\,\mathrm{Ai}'\,(\xi \exp [-2\pi i/3]) - \mathrm{Ai}'\,(\xi \exp [2\pi i/3])\,\mathrm{Ai}\,(\xi \exp [-2\pi i/3]) = i/2\pi. \quad \text{(A11)}$$

Prime denotes differentiation with respect to ξ in (A3), (A6), (A9), and (A11).

ACKNOWLEDGMENT

The authors wish to thank Dr. D. J. Mitchell and Dr. C. Pask for providing many helpful discussions and A. Ankiewicz for assistance throughout much of the early stages.

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